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A reliable treatment of Abel's second kind singular integral equations

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ABSTRACT

The central idea of this paper is to construct a new mechanism for the solution of Abel's type singular integral equations that is to say the two-step Laplace decomposition algorithm. The two-step Laplace decomposition algorithm (TSLDA) is an innovative adjustment in the Laplace decomposition algorithm (LDA) and makes the calculation much simpler. In this piece of writing, we merge the Laplace transform and decomposition method and present a novel move toward solving Abel's singular integral equations.

1. Introduction

Abel's Integral equations occur in the mathematical modeling of several models in physics, astrophysics, solid mechanics and applied sciences. Norwegian mathematician Niels Abel, gave the initiative of integral equations in 1823 in his study of mathematical physics [1,2]. Due to its wide-ranging importance and worth, Abel's Integral equations have been investigated by many researchers. Different techniques such as the Adomian decomposition method [3], the Homotopy perturbation method [4–6] and the variational iterative method [7] have been proposed for obtaining the approximate analytic solution of Abel's Integral equations.

At the present time, the decomposition method is extremely well known to researchers in finding solutions to different problems and many researchers make new amendments in the decomposition method of which the Laplace decomposition algorithm is one, introduced by Khuri [8,9]. The Laplace decomposition algorithm is applied extensively in applied physical problems efficiently [10–12]. Recently, a variation of LDA was proposed by different analytical experts [13]. The modified decomposition algorithm simply requires a minor variation from the standard LDA and has been shown to be computationally efficient but it has a problem of choosing a proper initial guess. The TSLDA overcomes this difficulty and explains how we can choose the initial approximation correctly without having noise terms.

2. Formulation of two-step Laplace decomposition method for Abel's singular integral equations

Let us consider Abel's singular integral equation of the form [1,2]

$$v(x) = \delta(x) + \int_0^x \beta(x, p)v(p)dp = \delta(x) + \int_0^x \frac{v(p)}{\sqrt{x-p}}dp, \tag{2.1}$$

where $\delta(x)$ is a non-homogeneous component and $\beta(x, p)$ is a kernel of the integral equation with singular behavior at $x = p$.

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On utilizing the Laplace transformation on both sides of Eq. (2.1), the yield is

$$£\left[v(x)\right] = £\left[\delta(x)\right] + £\left[\int_0^x \frac{v(p)}{\sqrt{x-p}} dp\right]. \tag{2.2}$$

Employing the convolution property of the Laplace transform Eq. (2.2) takes the form

$$£\left[v(x)\right] = £\left[\delta(x)\right] + £\left[\frac{1}{\sqrt{x}}\right] £\left[v(x)\right]. \tag{2.3}$$

Now implementing the inverse transformation on both sides of Eq. (2.3), we acquire

$$v(x) = \delta(x) + £^{-1}\left[\frac{\sqrt{\pi}}{\sqrt{s}} £\left[v(x)\right]\right]. \tag{2.4}$$

The Laplace decomposition algorithm supposes that the solution v is decomposed into infinite series as follows:

$$v = \sum_{m=0}^{\infty} v_m. \tag{2.5}$$

Making use of Eq. (2.5) in Eq. (2.4) we get

$$\sum_{m=0}^{\infty} v_m(x) = \delta(x) + £^{-1}\left[\frac{\sqrt{\pi}}{\sqrt{s}} £\left[\sum_{m=0}^{\infty} v_m\right]\right]. \tag{2.6}$$

The iterative relation takes the form

$$\begin{aligned} &v_0(x) = \delta(x), \\ &v_{i+1}(x) = £^{-1}\left[\frac{\sqrt{\pi}}{\sqrt{s}} £\left[v_i\right]\right], \quad i \geq 0. \end{aligned} \tag{2.7}$$

where $\delta(x)$ corresponds to the source term. Now we demonstrate TSLDA; after employing the inverse operator, we have $\delta(x)$ which is symbolized by the function χ as follow:

$$\chi = \delta(x). \tag{2.8}$$

By means of TSLDA, we put Eq. (2.8) in the following form

$$\chi = \delta_0(x) + \delta_1(x) + \delta_2(x) + \cdots + \delta_i(x), \tag{2.9}$$

where $\delta_0,\ \delta_1,\ \delta_2, \ldots, \delta_i$, are the terms that occur after applying the inverse Laplace transform on the source term $\delta(x)$. We define

$$v_0 = \delta_j(x) + \cdots + \delta_{j+s}(x), \tag{2.10}$$

where $j = 0,\ 1, \ldots, m, s = 0,\ 1, \ldots, m-k$. We authenticate that u_0 satisfies the original Eq. (2.1) and by substituting, once the exact solution is attained we are finished. Otherwise, we go to the next step. In this step, we place $u_0 = f(x)$ and start the iteration with the standard LDA i.e.;

$$v_{i+1}(x) = £^{-1}\left[\frac{\sqrt{\pi}}{\sqrt{s}} £\left[v_i\right]\right], \quad i \geq 0. \tag{2.11}$$

3. Numerical implementation of the method

In this section, we furnish two examples in order to display the usefulness of TSLDA.

3.1. Example

Consider a second kind Volterra equation in terms of Abel's integral equation as given by [1]

$$v(x) = \sqrt{x} - \pi x + 2\int_0^x \frac{v(t)}{\sqrt{x-t}} dt. \tag{3.1}$$

Operate the Laplace transform and using the convolution theorem of the Laplace transform, we have

$$£\left[v(x)\right] = \frac{\Gamma(3/2)}{s^{\frac{3}{2}}} - \frac{\pi}{s^2} + 2\frac{\Gamma(1/2)}{s^{\frac{1}{2}}} £\left[v(x)\right]. \tag{3.2}$$

Utilizing the inverse Laplace transform on Eq. (3.2), we get

$$v(x) = \sqrt{x} - \pi x + 2£^{-1}\left[\frac{\Gamma(1/2)}{s^{\frac{1}{2}}}£[v(x)]\right]. \tag{3.3}$$

We decompose our solution into an infinite number of components The Laplace decomposition method decomposes the solution into an infinite number of components given as follows:

$$v = \sum_{n=0}^{\infty} v_n(x). \tag{3.4}$$

Invoking Eq. (3.4) in Eq. (3.3) yields

$$\sum_{n=0}^{\infty} v_n(x) = \sqrt{x} - \pi x + 2£^{-1}\left[\frac{\Gamma(1/2)}{s^{\frac{1}{2}}}£\left[\sum_{n=0}^{\infty} v_n(x)\right]\right]. \tag{3.5}$$

From Eq. (3.5), our mandatory recurrence relation is given below

$$v_0(x) = \sqrt{x} - \pi x, \tag{3.6}$$

$$v_{n+1}(x) = 2£^{-1}\left[\frac{\Gamma(1/2)}{s^{\frac{1}{2}}}£[v_n(x)]\right], \quad n \geq 0. \tag{3.7}$$

The first few components of $v_n(x)$ by using the recursive relation (3.7) as follows immediately gives

$$v_1(x) = \pi x - \frac{8\pi}{3}x^{3/2}, \tag{3.8}$$

$$\vdots$$

$$\vdots$$

As it can be observed from the original Eq. (3.1) the non-homogeneous Abel's second kind singular integral equation and the exact solution in the zeroth component v_0, there is the phenomenon of noise terms [6]. By investigating v_0 and v_1 we can simply observe the appearance of noise term πx in v_0. Consequently, by canceling the noise term in v_0, the terms left behind offer an exact solution.

Two-step Laplace decomposition algorithm

By using TSLDA, we decompose the function $\delta(x)$ as follows:

$$\delta_1(x) = \delta_{10}(x) + \delta_{11}(x), \tag{3.9}$$

where

$$\delta_{10}(x) = \sqrt{x}, \qquad \delta_{11}(x) = -\pi x, \tag{3.10}$$

where δ_{11} does not satisfy Eq. (3.1). By choosing $v_0 = \delta_{10}$ and by confirming that v_0 satisfy Eq. (3.1), the exact solution will be achieved straight away. Therefore, our proposed adjusted recursive relation becomes

$$v_0(x) = \sqrt{x}, \tag{3.11}$$

$$v_1(x) = -\pi x + 2£^{-1}\left[\frac{\Gamma(1/2)}{s^{\frac{1}{2}}}£[u_0(x)]\right] = 0, \tag{3.12}$$

$$\vdots$$

$$v_{n+1}(x) = 0, \quad n \geq 1. \tag{3.13}$$

Consequently, solution by TSLDA is

$$v = \sum_{n=0}^{\infty} v_n(x) = \sqrt{x}. \tag{3.14}$$

3.2. Example

Consider another second kind of Volterra singular integral equation given by [1]

$$v(x) = \frac{1}{2} - \sqrt{x} + \int_0^x \frac{v(p)}{\sqrt{x-p}} dp. \tag{3.15}$$

Using the Laplace transform and convolution property, Eq. (3.15) takes the following form

$$\pounds\,[v(x)] = \frac{1}{2}\frac{1}{s} + \frac{\Gamma(3/2)}{s^{\frac{3}{2}}} + \frac{\Gamma(1/2)}{s^{\frac{1}{2}}}\pounds\,[v(x)]\,. \tag{3.16}$$

Employing the inverse Laplace transform to Eq. (3.16), we have

$$v(x) = \frac{1}{2} - \sqrt{x} + \pounds^{-1}\left[\frac{\Gamma(1/2)}{s^{\frac{1}{2}}}\pounds\,[v(x)]\right]. \tag{3.17}$$

The Laplace decomposition method assumes that the solution function $v(x)$ can be decomposed as an infinite series as follow:

$$v = \sum_{n=0}^{\infty} v_n(x). \tag{3.18}$$

Using Eq. (3.18) in Eq. (3.17) yields

$$\sum_{n=0}^{\infty} v_n(x) = \frac{1}{2} - \sqrt{x} + \pounds^{-1}\left[\frac{\Gamma(1/2)}{s^{\frac{1}{2}}}\pounds\left[\sum_{n=0}^{\infty} v_n(x)\right]\right]. \tag{3.19}$$

From Eq. (3.19), our required recursive relation is given below

$$v_0(x) = \frac{1}{2} - \sqrt{x}, \tag{3.20}$$

$$v_{n+1}(x) = \pounds^{-1}\left[\frac{\Gamma(1/2)}{s^{\frac{1}{2}}}\pounds\,[v_n(x)]\right], \quad n \geq 0. \tag{3.21}$$

The first few components of $v_n(x)$ by using the recursive relation (3.21) as follows immediately gives

$$v_1(x) = \sqrt{x} - \frac{\pi x}{2}, \tag{3.22}$$

$$\vdots$$

$$\vdots$$

From the zeroth and first order components, we observe that the noise terms phenomenon appears in the solution while using the standard Laplace decomposition method.

Two-step Laplace decomposition algorithm

By using TSLDA, we decompose the function arising from the non-homogeneous part of the equation i.e.; $\delta_2(x)$ as follows

$$\delta_2(x) = \delta_{20}(x) + \delta_{21}(x), \tag{3.23}$$

where

$$\delta_{20}(x) = \frac{1}{2}, \qquad \delta_{21}(x) = -\sqrt{x}. \tag{3.24}$$

It is obvious that $\delta_{21}(x)$ does not satisfy Eq. (3.15). By choosing $v_0 = \delta_{20}(x)$ and by verifying that v_0 satisfies Eq. (3.15), the exact solution will be obtained immediately and we have

$$\begin{aligned} v_0(x) &= \frac{1}{2}, \\ v_1(x) &= -\sqrt{x} + \pounds^{-1}\left[\frac{\Gamma(1/2)}{s^{\frac{1}{2}}}\pounds\,[v_0(x)]\right] = 0, \\ &\vdots \\ v_{n+1}(x) &= 0, \quad n \geq 1. \end{aligned} \tag{3.25}$$

Therefore, the solution by TSLDA is

$$v = \sum_{n=0}^{\infty} v_n(x) = \frac{1}{2}. \tag{3.26}$$

which is the exact solution [1] of Eq. (3.15).

4. Concluding remarks

In the present work, first we developed a solution procedure. We have solved two examples with our proposed scheme. We observe that our developed mechanism is straightforward and easy to apply. We can easily find a solution of Abel type singular integral equations within two steps. This serves as a fundamental method in solving singular integral equations.

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